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LONGITUDINAL WAVES IN GRAINY MEDIA

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Based on a hypoplastic model of media with dilatancy, a nonlinear inhomogeneous wave equation describing compression waves in soil is derived, and its solution is analyzed.

The study of propagation laws of seismic waves in soil allows one to obtain more complete information on mechanical properties of grainy media. The construction of soil models is generally based on the concepts of mechanics of continuous media, which implies averaging of the kinematic and dynamic characteristics of grains over physically infinitesimal volumes, i.e., volumes that are sufficiently small in comparison with characteristic spatial scales of the processes studied but contain a large number of grains. The balance law of momentum for all models is written identically:

$$\rho \dot{v}_i = \frac{\partial \sigma_{ij}}{\partial x_j} \qquad (i,\, j = 1,\, 2,\, 3).$$

Here ρ is the medium density, v_i is the component of the velocity vector $\boldsymbol{v}$, σ_{ij} is the stress tensor, and x_i are the Cartesian coordinates; the dot indicates the material derivative $d/dt = \partial/\partial t + v_j \partial/\partial x_j$; summation is performed over repeated indices. It is necessary to introduce a dependence between stresses and strains (or strain rates), in addition to the above equations, in order to obtain a closed mathematical model. Such coupling is known as a constitutive equation or a closing relation.

The simplest constitutive equation is the Hooke's law $\sigma = \sigma(\varepsilon)$. It is seen from this equation that the stress is independent of the loading path and deformation history. Materials and media that can be described by this kind of relations are called elastic. They are studied by the theory of elasticity. Numerous field and laboratory observations show that soils are not elastic; therefore, closing relations should be different.

In early 1950s, Truesdell introduced a constitutive equation in the form of a dependence between instantaneous stresses and strains, which is written as an evolution equation relating the stress rates and strain rates in the medium $d\sigma = f(d\varepsilon)$ [or $\dot{\sigma} = f(\dot{\varepsilon})$, where $\dot{\varepsilon}_{ij} \equiv D_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ is the stretching tensor]. The derivative $d\sigma/d\varepsilon$ characterizes the stiffness of the material considered. Soils are inelastic (plastic) media, and their stiffness is much higher under unloading than under loading. The closing relations derived by Trusdell and called hypoelastic may be written in the generic form $\dot{\sigma} = h(\sigma, D)$, where the function h is linear in terms of the functions σ and D. A medium is hypoelastic if the stress-rate tensor is a linear function of the stress-rate tensor at every point and every moment of time. This function, in turn, may depend linearly on the stress tensor [1, 2].

Constructing the model usually involves the derivative of the stress tensor in time, which vanishes when the material, as a solid body, rotates in a fixed coordinate system. Such a derivative is named the Jaumann derivative and has the form

$$\sigma_{ij}^0 = \frac{d\sigma_{ij}}{dt} + (\sigma\Omega)_{ij} - (\Omega\sigma)_{ij},$$

where $\Omega_{ij} = (\partial v_i/\partial x_j - \partial v_j/\partial x_i)/2$ is the spin tensor. Thus, a hypoelastic medium is described by the following incremental law:

$$\frac{d\sigma_{ij}}{dt} = -(\sigma\Omega)_{ij} + (\Omega\sigma)_{ij} + L_{ijkl}(\sigma) D_{ij}.$$

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Here L_{ijkl} is a tensor function, which, in the general case, may linearly depend on the components of the stress tensor. For the simplest hypoelastic isotropic medium, this function can be written in the form $L_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ are the Lamé coefficients.

It follows from the last equation that the stiffness of a material is the same both under loading and unloading. Therefore, the constitutive equations for a hypoelastic medium are not suitable for the description of soils, which possess the property of plasticity and whose stiffness is smaller under loading than under unloading. Sometimes, this problem can be solved using two different relations between $\dot{\sigma}$ and $\dot{\varepsilon}$, one of them is chosen for sections of medium loading and the second one is for sections of unloading. As an example, we can write closing relations in the one-dimensional case ($u = v_1$ and $x = x_1$) for small loading and unloading of the elastic medium in finite increments $\Delta\sigma = \mu_s\Delta\varepsilon$ or in the differential form $\sigma_t = \mu_s u_x$, where the subscript x denotes differentiation over the coordinate, the stiffness is $\mu_s = \mu_1$ under loading ($\Delta\sigma < 0$) or $\mu_s = \mu_2$ under unloading ($\Delta\sigma > 0$), where $\mu_2 > \mu_1$. If we denote $\nu_1 = (\mu_2 + \mu_1)/2$ and $\nu_2 = (\mu_2 - \mu_1)/2$, then the above differential equations can be presented in the form of one equation $\sigma_t = \nu_1 u_x + \nu_2|u_x|$, which is valid both for loading and unloading. This equation contains the absolute value of the derivative of the medium velocity over the coordinate and because of this, it is nonlinear even in the small. We note that the procedure described is known in computational mathematics and is used for formal transition from two-point to three-point grid equations with further application of the sweep method.

This approach is currently used to describe grainy media, in particular, sands [1–5]. The constitutive equation of a hypoelastic medium can be written in the general form as

$$\frac{d\sigma_{ij}}{dt} = -(\sigma\Omega)_{ij} + (\Omega\sigma)_{ij} + L_{ijkl}(D,e)D_{ij} + N_{ij}(D,e)\|D\|,$$

where $e = (V - V_s)/V_s$ is the porosity of the medium (V_s is the volume of the solid phase and V is the total volume), $\|D\| = \sqrt{\operatorname{tr}(D^2)}$ is the norm of the strain-rate tensor, and L and N are some tensor functions of the indicated arguments, which have different presentations including constants that characterize the material properties.

To study compression waves in grainy media, we chose the mathematical model [4], in which the changes in porosity are neglected. This model includes the equations of motion and incremental coupling of stresses and strains

$$\rho\,\frac{dv_i}{dt} = \frac{\partial\sigma_{ij}}{\partial x_j}; \tag{1}$$

$$\frac{d\sigma_{ij}}{dt} = -(\sigma\Omega)_{ij} + (\Omega\sigma)_{ij} + f_1(\sigma)D_{ij} + f_2(\sigma)\sigma_{ij}\operatorname{tr}(\sigma D) + \varphi_{ij}\|D\|. \tag{2}$$

Here $f_1(\sigma) = C_1\operatorname{tr}(\sigma)$, $f_2(\sigma) = C_2/\operatorname{tr}(\sigma)$, $\varphi_{ij}(\sigma) = (C_3(\sigma^2)_{ij} + C_4(\sigma^{*2})_{ij})/\operatorname{tr}(\sigma) + (C_5(\sigma^3)_{ij} + C_6(\sigma^{*3}_{ij}))/\operatorname{tr}(\sigma^2)$, and $\sigma^*_{ij} = \sigma_{ij} - (1/3)\operatorname{tr}(\sigma)\delta_{ij}$ ($C_1, \ldots, C_6$ are empirical constants determined from test experiments in combination with solving the constitutive equation). This procedure is called calibration.

The presence of the term proportional to the norm of the strain-rate tensor in (2) makes the considered hypoelastic model nonlinear even in the small, because it does not admit linearization in the vicinity of $\|D\| = 0$.

Let us examine one-dimensional longitudinal movements of the hypoelastic media. We assume that the sought functions depend only on one coordinate, for example, x, and on the time t. We can consider that stress deviations from the initial values are small ($\|\sigma - \sigma^0\| \ll \|\sigma^0\|$), the axes x, y, and z coincide with the main axes of the stress tensor, and the initial stressed state σ^0 is uniform and hydrostatic. The velocity vector has only the x-component: $\boldsymbol{v} = (u, 0, 0)$. Then, we obtain the following equation for the longitudinal component of the material velocity of the medium:

$$u_{tt} - c_p^2 u_{xx} = b|u_x|_x. \tag{3}$$

Here, the coefficient c_p, which has the dimension of velocity and depends on the undisturbed stressed state, is determined by the formula $c_p^2 = (C_1\operatorname{tr}(\sigma^0) + C_2(\sigma^0_{xx})^2/\operatorname{tr}(\sigma^0))/\rho$ and may be called the velocity of longitudinal waves. The coefficient b, which has the dimension of velocity squared, characterizes the nonlinear and dilatant properties of the medium, depends on the characteristics of the undisturbed state, and equals

$$b = (1/\rho)\{(\sigma^0_{xx} + \sigma^0_{yy} + \sigma^0_{zz})^{-1}[C_3(\sigma^0_{xx})^2 + C_4(\sigma^0_{xx} - \operatorname{tr}(\sigma^0)/3)^2]$$

$$+ [(\sigma^0_{xx})^2 + (\sigma^0_{yy})^2 + (\sigma^0_{zz})^2]^{-1}[C_5(\sigma^0_{xx})^3 + C_6(\sigma^0_{xx} - \operatorname{tr}(\sigma^0)/3)^3]\}.$$

This coefficient can be positive as well as negative depending on the initial stress σ^0 and on the model constants $C_3, \ldots, C_6$.

Equation (3) is a uniform wave equation with a nonlinear source. Let us assume that the undisturbed stressed state is isotropic: $\sigma^0_{xx} = \sigma^0_{yy} = \sigma^0_{zz} \equiv \sigma_0$. Then, the expressions for the longitudinal-wave velocity and the parameter b are significantly simplified: $c_p^2 = (3C_1 + C_2/3)\sigma_0/\rho$ and $b = (C_3 + C_5)\sigma_0/(3\rho)$. Let us study the properties of compression waves. We factorize the wave operator

$$\frac{\partial^2}{\partial t^2} - c_p^2 \frac{\partial^2}{\partial x^2} = \Big(\frac{\partial}{\partial t} - c_p \frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + c_p \frac{\partial}{\partial x}\Big)$$

and choose a wave propagating in the positive direction of the x axis, using the approximate approach described, for example, in [6]. For an arbitrary wave profile propagating toward $x > 0$ with a velocity approximately equal to U, the derivatives over x and t are coupled by the relation $\partial/\partial t \approx -U\partial/\partial x$. This may be used to analyze the features of wave packets consisting of waves propagating with different velocities. To identify the wave moving in the positive direction of the x axis and described by the operator $\partial/\partial t + c_p\partial/\partial x$, we perform the substitution $\partial/\partial t \approx -c_p\partial/\partial x$ in the operator $\partial/\partial t - c_p\partial/\partial x$, which corresponds to the wave propagating in the direction $x < 0$. As a result, we obtain the equation

$$-2c_p \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial t} + c_p \frac{\partial u}{\partial x}\Big) = b \frac{\partial}{\partial x}\Big|\frac{\partial u}{\partial x}\Big|.$$

Integration of this equation over the coordinate yields the first-order equation for the longitudinal wave:

$$u_t + c_p u_x + \frac{b}{2c_p}\,|u_x| = 0. \tag{4}$$

The presence of the modulus of the material-velocity derivative over the coordinate makes this equation nonlinear even in the small. Analyzing Eq. (4), we can note that, if we have $u_x > 0$ everywhere, then the indicated equation takes the form $u_t + (c_p + b/(2c_p))u_x = 0$, which corresponds to the transfer of the sought function in the positive direction of the x axis with a velocity $c_1 = c_p + b/(2c_p)$. If we have $u_x < 0$ everywhere, then Eq. (4) reduces to the form $u_t + (c_p - b/(2c_p))u_x = 0$, which corresponds to the transfer of the sought function in the direction $x > 0$ with a velocity $c_2 = c_p - b/(2c_p)$.

The initial condition for the numerical solution of Eq. (4) is the spatially localized distribution

$$u(x, 0) = u_0 \exp\,[-(x - x_0)^2/l^2]; \tag{5}$$

the boundary conditions are accepted in the form

$$\lim_{x\to\pm\infty} u(x, t) = 0. \tag{6}$$

This pulse is symmetric with respect to the point x_0. The derivative u_x of the initial function (5) changes its sign at $x = x_0$, Therefore, the perturbation profile changes with time, because sections with a positive derivative move with the velocity c_1 and those with a negative derivative, with the velocity $c_2 > c_1$. The solution in Fig. 1 shows that the perturbation amplitude decreases with time (the greater the parameter b, the faster the decrease in amplitude), and the pulse completely decays. If we change the sign of this parameter, the evolution character also changes: profile sections with a positive derivative propagate with the velocity $c_1' = c_p - |b|/(2c_p)$ and those with a negative derivative, with the velocity $c_2' = c_p + |b|/(2c_p)$, where $c_2' > c_1'$. Therefore, the fore front of the perturbation, where $u_x < 0$, leads the rare front, where $u_x > 0$, and the pulse expands, retaining the amplitude unchanged (Fig. 2).

Let us consider the solutions of the wave equation (3), which describes the propagation of compression waves both in the positive and negative directions of the x axis. First, changing the sign of the coordinate $x \to -x$ is equivalent to changing the sign of the parameter $b \to -b$. Assuming that $b > 0$, we obtain $c_1 - c_2 = b/c_p > 0$, whence it follows that the sections of the perturbation profile with $u_x > 0$ propagate faster than the sections with $u_x < 0$. This should lead to distortion of the profiles of compression waves propagating over the medium. We also note that Eq. (3) remains unchanged with simultaneous changing of the signs $x \to -x$ and $b \to -b$. Therefore, the solutions corresponding to negative values of the parameter b are the mirror reflection around the points $x = x_0$ of the solutions corresponding to positive values of this coefficient. Let us choose an initial disturbance in the form

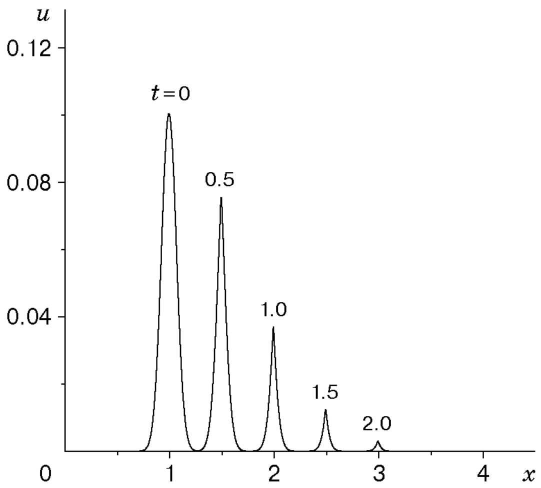

Fig. 1

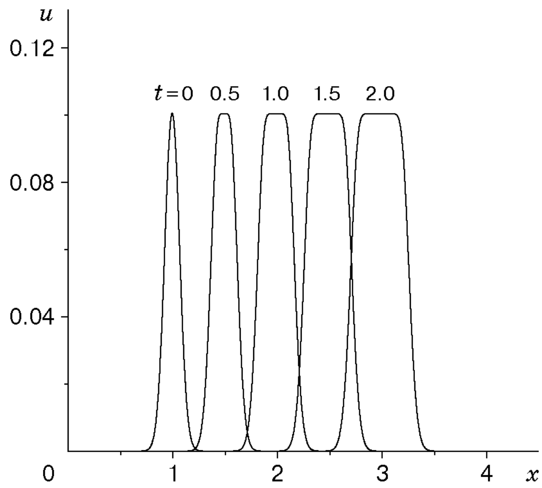

Fig. 2

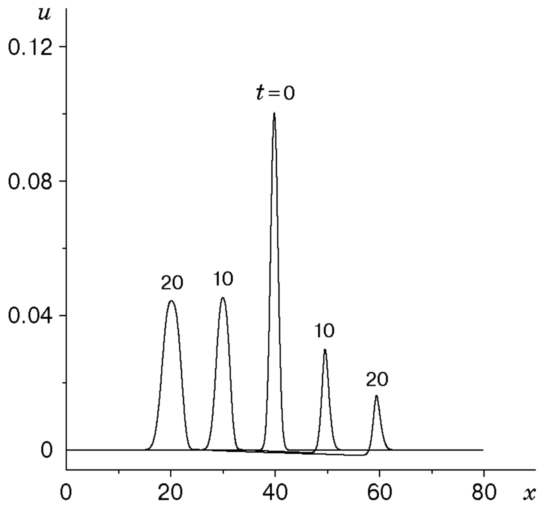

Fig. 3

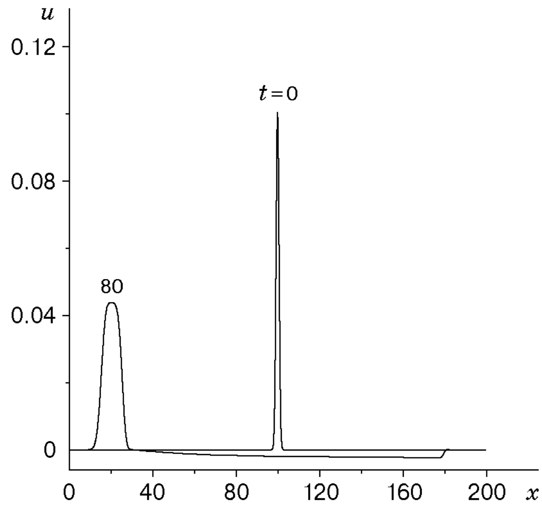

Fig. 4

of a spatially localized pulse (5), setting additionally $u_t(x, 0) = 0$ and the boundary conditions (6). The condition $u_t(x, 0) = 0$ corresponds to the choice of the initial stress disturbance σ_{xx} satisfying the condition $\partial\sigma_{xx}(x, 0)/\partial x = 0$ for all values of x. It follows from the calculations that such an initial perturbation generates wave motion in both directions of the x axis, whose patterns are not symmetrical (Fig. 3). The parameter b is chosen positive; thus, for comparatively small times, the pulse moving left changes as it follows from the transfer equation (4) for $b < 0$: the pulse amplitude remains constant, and its width increases. The amplitude of the pulse moving to the right decreases tending to zero. At large times, only the pulse moving to the left remains, which is accompanied by the "tail" of a small amplitude with the opposite sign, which expands to the right (Fig. 4). The further evolution can be investigated only using nonlinearalized equations (1), (2).

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